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POLE–ZERO CANCELLATION IN STRUCTURES: REPEATED ROOTS

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The conditions for the creation of nodes of normal modes of vibration from the cancellation of poles and zeros are established when either the poles or the zeros (or both) appear as repeated eigenvalues. The analysis is illustrated by numerical examples including the case of a pole–zero cancellation at every co-ordinate at the same frequency which is shown to occur whenever there are repeated poles. If there are repeated poles and repeated zeros at the same frequency then the number of poles must be either one more, one less or equal to the number of zeros.

1. INTRODUCTION

There is interest in manipulating the nodes of normal modes of vibration mainly for two reasons: (i) to protect sensitive equipment from damage by siting it at a node—this might involve shifting the node spatially because of physical constraints; and (ii) to desensitize a part that is less well understood than the rest of the structure—this can bring about an improvement in the capability of a mathematical model to represent the dynamics of a physical structure. Vibration nodes are created by the mutual cancellation of a pole (natural frequency) with a zero (antiresonance). Mottershead and Lallement [1] established the necessary and sufficient conditions for the creation of a vibration node by the cancellation of a pole with a distinct zero. Mottershead [2] studied the sensitivities of the zeros.

There are four main categories of modification methods for shifting poles and zeros. The unit-rank modification approach [1, 3, 4] has the advantage that the natural frequencies of a modified structure can be inferred from receptances obtained from the structure in its unmodified condition. The more general methods [5–7] require the adjustment of several mass and stiffness terms and are related to

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the problem of finite element model updating [8, 9]. Cha and Pierre [10] used a chain of mass–spring oscillators to impose a node either at the point of connection (collocated) or elsewhere on the structure (uncollocated), and Ram and Elhay [11] studied the multi-degree-of-freedom dynamic absorber. The fourth category is that of pole–zero assignment by using active control techniques [12, 13].

This paper addresses the problem of pole–zero cancellation when there are either repeated poles or repeated zeros (or both) present in the measured frequency responses. These cases were not considered in reference [1] and their investigation leads to to an understanding of how nodes are created in the presence of multiple roots (either zeros or poles). In a numerical example it is shown how a pole–zero cancellation at every co-ordinate at the same frequency can produce an apparently lower order system than the dimension of the mass and stiffness matrices. This phenomenon is shown to occur whenever there are repeated poles.

2. COINCIDENT POLES AND ZEROS

When the stiffness and mass matrices, $\mathbf{K}, \mathbf{M} \in \Re^{n \times n}$, $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > 0$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}} \geqslant 0$ (or > 0), are partitioned so as to separate a co-ordinate then, choosing the first co-ordinate without loss of generality, the matrices can be written as

$$\mathbf{K} = \left[\begin{array}{c|c} k_{11} & \bar{\mathbf{k}}^{\mathrm{T}} \\ \hline \bar{\mathbf{k}} & \bar{\mathbf{K}} \end{array}\right], \qquad \mathbf{M} = \left[\begin{array}{c|c} m_{11} & \bar{\mathbf{m}}^{\mathrm{T}} \\ \hline \bar{\mathbf{m}} & \bar{\mathbf{M}} \end{array}\right]. \tag{1, 2}$$

The transformation matrix

$$\mathbf{R} = \left[\begin{array}{c|c} 1 & \\ \hline & \boldsymbol{\Psi} \end{array}\right], \tag{3}$$

$$\boldsymbol{\Psi}^{\mathrm{T}}\bar{\mathbf{K}}\boldsymbol{\Psi} = diag(\kappa_i), \qquad \boldsymbol{\Psi}^{\mathrm{T}}\bar{\mathbf{M}}\boldsymbol{\Psi} = diag(\mu_i) \tag{4, 5}$$

may be applied to produce

$$\mathbf{A} = \left[\begin{array}{c|c} k_{11} & \mathbf{a}^{\mathrm{T}} \\ \hline \mathbf{a} & diag(\kappa_i) \end{array}\right], \tag{6}$$

$$\mathbf{B} = \left[\begin{array}{c|c} m_{11} & \mathbf{b}^{\mathrm{T}} \\ \hline \mathbf{b} & diag(\mu_i) \end{array}\right], \tag{7}$$

$$\mathbf{a}^{\mathrm{T}} = \bar{\mathbf{k}}^{\mathrm{T}}\boldsymbol{\Psi}, \quad \mathbf{b}^{\mathrm{T}} = \bar{\mathbf{m}}^{\mathrm{T}}\boldsymbol{\Psi}, \tag{8, 9}$$

where $\mathbf{B}^{-1}\mathbf{A}$ is similar to $\mathbf{M}^{-1}\mathbf{K}$. Then by expanding the determinant

$$det(\mathbf{A} - \lambda_r \mathbf{B}) = 0, \tag{10}$$

one obtains an expression previously derived by Mottershead and Lallement [1] which gives the rth pole λ_r in terms of the zeros $\bar{\lambda}_i = \kappa_i/\mu_i$, $i = 1, \ldots, n-1$, of the point receptance h_{11},

$$(k_{11} - \lambda_r m_{11}) \prod_{i=1}^{n-1} (\kappa_i - \lambda_r \mu_i) - \sum_{j=1}^{n-1} (a_j - \lambda_r b_j)^2 \prod_{\substack{i=1 \\ i \neq j}}^{n-1} (\kappa_i - \lambda_r \mu_i) = 0. \tag{11}$$

Certain physical insights for the pole–zero cancellation problem ($\lambda_r = \bar{\lambda}_s$) can be obtained from equation (11) and are best appreciated when it is re-written in full as

$$\begin{aligned}
&(k_{11} - \lambda_r m_{11})(\kappa_1 - \lambda_r \mu_1) \cdots (\kappa_s - \lambda_r \mu_s) \cdots (\kappa_{n-1} - \lambda_r \mu_{n-1}) \\
&- (a_1 - \lambda_r b_1)^2 (\kappa_2 - \lambda_r \mu_2) \cdots (\kappa_s - \lambda_r \mu_s) \cdots (\kappa_{n-1} - \lambda_r \mu_{n-1}) \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&- (a_s - \lambda_r b_s)^2 (\kappa_1 - \lambda_r \mu_1) \cdots (\kappa_{s-1} - \lambda_r \mu_{s-1})(\kappa_{s+1} - \lambda_r \mu_{s+1} \cdots (\kappa_{n-1} - \lambda_r \mu_{-1}) \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&- (a_{n-1} - \lambda_r b_{n-1})^2 (\kappa_1 - \lambda_r \mu_1) \cdots (\kappa_s - \lambda_r \mu_s) \cdots (\kappa_{n-2} - \lambda_r \mu_{n-2}) = 0.
\end{aligned} \tag{12}$$

When the sth zero $\bar{\lambda}_s$ is distinct it is clear that the term $(\kappa_s - \lambda_r \mu_s)$, which occurs as a multiplier in all components but one of the sum in equation (12), will go to zero. Then, since $(\kappa_i - \lambda_r \mu_i) \neq 0$, $i = 1, \ldots, n-1$, $(i \neq s)$, it follows that

$$(a_s - \lambda_r b_s) = 0. \tag{13}$$

By combining equations (13), (8) and (9) it is found that when the zeros are distinct then a pole–zero cancellation $\lambda_r - \bar{\lambda}_s$ always brings about the relationship

$$(\bar{\mathbf{k}} - \bar{\lambda}_s \bar{\mathbf{m}})^{\mathrm{T}} \boldsymbol{\psi}_s = \mathbf{0}. \tag{14}$$

From equations (1), (2) and (14) it is apparent that the eigenvalue problem of the distinct zeros can be written as

$$\left(\begin{bmatrix} -\bar{\mathbf{k}}^{\mathrm{T}}- \\ \bar{\bar{\mathbf{K}}} \end{bmatrix} - \bar{\lambda}_s \begin{bmatrix} -\bar{\mathbf{m}}^{\mathrm{T}}- \\ \bar{\bar{\mathbf{M}}} \end{bmatrix} \right) \boldsymbol{\psi}_s = \mathbf{0}. \tag{15}$$

Since the zeros of the cross-receptance h_{jk} are given by $\bar{\lambda}_i(\mathbf{K}, \mathbf{M})_{jk}$, $i = 1, \ldots, n-1$, where the subscripts denote the deletion of the jth row and kth column of $\mathbf{K}$ and $\mathbf{M}$, then the pole–zero cancellation that occurs in the point receptance h_{11} must also occur in all the cross receptances of the first co-ordinate. Such a cancellation can be recognized as a node of the rth normal mode because (i) if an input is applied at a node then the mode will not be excited anywhere in the structure, and (ii) a measurement at a node will exclude the mode regardless of where the input is applied. By comparing equation (15) with the eigenvalue problem of the poles,

$$(\mathbf{K} - \lambda_r \mathbf{M})\boldsymbol{\varphi}_r = \mathbf{0}, \tag{16}$$

it is seen that

$$\boldsymbol{\varphi}_r = \left\{\frac{0}{\boldsymbol{\psi}_s}\right\}, \tag{17}$$

which shows that the first co-ordinate is indeed a vibration node. Since $\boldsymbol{\psi}_s$ spans *null* $\left(\left[\frac{\bar{\mathbf{k}}^{\mathrm{T}}}{\bar{\mathbf{K}}}\right] - \bar{\lambda}, \left[\frac{\bar{\mathbf{m}}^{\mathrm{T}}}{\bar{\mathbf{M}}}\right]\right)$ then $\left(\frac{0}{\boldsymbol{\psi}_s}\right)$ must span $null(\mathbf{K} - \lambda_r \mathbf{M})$, $\lambda_r - \bar{\lambda}_s$, when either λ_r is distinct or $\boldsymbol{\varphi}_r = \left(\frac{0}{\boldsymbol{\psi}_s}\right)$ is a combination of eigenvectors of repeated poles. It was shown by Mottershead and Lallement [1] that $\lambda_r = \bar{\lambda}_s$ is a necessary condition and equation (14) is a sufficient one for the creation of a vibration node. Furthermore, when the coincident zero is distinct a cancellation is impossible unless equation (14) is satisfied. In the sequel, we consider how vibration nodes are formed when there are repeated zeros in the measured receptances, and how repeated poles will always give rise to pole–zero cancellations at every co-ordinate.

3. VIBRATION NODES FROM REPEATED ZEROS

In the case of a repeated zero of multiplicity $m + 1$, $\bar{\lambda}_s = \bar{\lambda}_{s+t}$, $t = 1, \ldots, m(m + 1 \leqslant n - 1)$, every component of the sum in equation (12) will independently go to zero without fulfilling the sufficient condition (14). It will be demonstrated that this has no effect on the well-known characteristic of vibration nodes, that a pole–zero cancellation in a point receptance h_j is accompanied by a cancellation in every cross receptance $h_{jk} = h_{kj}$, $k \neq j$. Consider the eigenvalue equation of the zeros,

$$(\bar{\mathbf{K}} - \bar{\lambda}_s \bar{\mathbf{M}})\boldsymbol{\psi}_s = \mathbf{0}, \quad \bar{\lambda}_s = \lambda_r, \tag{18}$$

and also the eigenvalue equation of the poles (but omitting the first row),

$$([\bar{\mathbf{k}}\,|\,\bar{\mathbf{K}}] - \lambda_r[\bar{\mathbf{m}}\,|\,\bar{\mathbf{M}}])\boldsymbol{\varphi}_r = \mathbf{0}. \tag{19}$$

Then since $(\bar{\mathbf{k}} - \lambda_r \bar{\mathbf{m}}) \neq \mathbf{0}$, there exists a solution

$$\boldsymbol{\varphi}_{1r} = 0 \quad \text{and} \begin{Bmatrix} \boldsymbol{\varphi}_{2r} \\ \vdots \\ \boldsymbol{\varphi}_{nr} \end{Bmatrix} = \boldsymbol{\psi}_s.$$

We now consider how a cancellation that fails to satisfy equation (14) may produce a node.

The eigenvectors of the repeated zeros must span the null space of $(\bar{\mathbf{K}} - \bar{\lambda}_s \bar{\mathbf{M}})$. So,

$$null(\bar{\mathbf{K}} - \bar{\lambda}_s \bar{\mathbf{M}}) = (\boldsymbol{\psi}_s, \boldsymbol{\psi}_{s+1}, \ldots, \boldsymbol{\psi}_{s+m}), \quad 1 < m \leqslant n-1, \tag{20}$$

and any vector of the form

$$[\boldsymbol{\psi}_s, \boldsymbol{\psi}_{s+1}, \ldots, \boldsymbol{\psi}_{s+m}]\boldsymbol{\alpha}$$

will be an eigenvector. It follows that if and only if

$$(\bar{\mathbf{k}} - \bar{\lambda}_s \bar{\mathbf{m}}) \perp [\boldsymbol{\psi}_s, \boldsymbol{\psi}_{s+1}, \ldots, \boldsymbol{\psi}_{s+m}]\boldsymbol{\alpha}, \tag{21}$$

and

$$\bar{\lambda}_s = \lambda_r, \tag{22}$$

will a vibration node be created at the first co-ordinate ($\varphi_{1r} = 0$). Equation (21) is a more general form of the sufficient condition (14). It means that there must be a zero $\bar{\lambda}_s$ in the point receptance h_j and in every cross receptance h_{jk} to cancel with the pole λ_r and produce a node at the jth co-ordinate. It is impossible for φ_{1r} to take any other value than zero if λ_r is distinct.

3.1. NUMERICAL EXAMPLE

When all the stiffnesses (except $k_8 = \frac{5}{3}$) and all the masses of the system in Figure 1 are unity the poles and zeros (of receptance h_{44}) take the values given in Table 1. A pole and two zeros coincide at 2 rad/s. The eigenvectors of the two zeros are listed in Table 2. The row $(\bar{\mathbf{k}} - \bar{\lambda}_s \bar{\mathbf{m}})^{\mathrm{T}} = (0\ 0\ -1\ -1\ 0)$ so that the vector $\boldsymbol{\alpha}$ in equation (21) can be determined within an arbitrary scalar multiplier to be $\boldsymbol{\alpha} = (0{\cdot}9186,\ -0{\cdot}3953)^{\mathrm{T}}$. There is no other combination of the vectors $\boldsymbol{\psi}_4$ and $\boldsymbol{\psi}_5$ that will satisfy equation (21). This means that only one of the two zeros can cancel with the pole. The point receptance h_{44} is plotted in Figure 2 where a single zero (the uncancelled one) is shown to exist at 2 rad/s. Since the eigenvector of the uncancelled zero fails to satisfy equation (14) this zero cannot exists in the cross-receptances, such as h_{34} which is shown in Figure 3. There is, however, evidence of the pole–zero cancellation in Figure 3 which shows only five poles.

4. VIBRATION NODES FROM REPEATED POLES

In the case of repeated poles of multiplicity $p+1$, $\lambda_r = \lambda_{r+q}$, $q = 1, \ldots, p(p + 1 \leqslant n)$, the eigenvalue equation can be written as

$$(\mathbf{K} - \lambda_r \mathbf{M})[\boldsymbol{\varphi}_r\, \boldsymbol{\varphi}_{r+1}, \ldots, \boldsymbol{\varphi}_{r+p}]\boldsymbol{\beta} = 0. \tag{23}$$

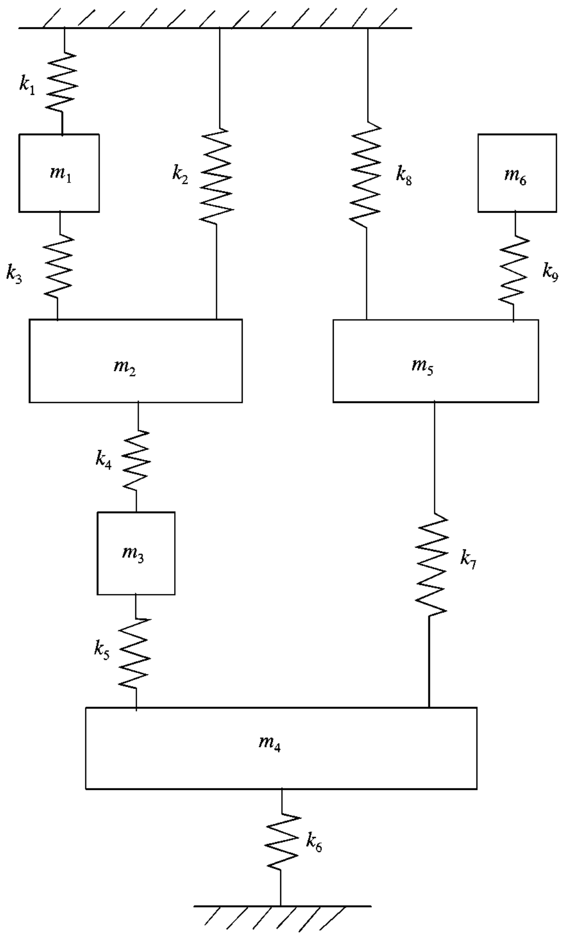

Figure 1. Six-degree-of-freedom mass–spring system.

TABLE 1

Table of poles and zeros

Frequency (rad/s)	
Pole	Zero
0·7530	0·8165
0·9129	1·0000
1·2896	1·4142
1·6923	2·0000
2·0000	2·0000
2·1770	

TABLE 2

Eigenvectors of the coincident zeros

Eigenvectors	
ψ_4	ψ_5
0·4082	0
− 0·8165	0
0·4082	0
0	0·9487
0	− 0·3162

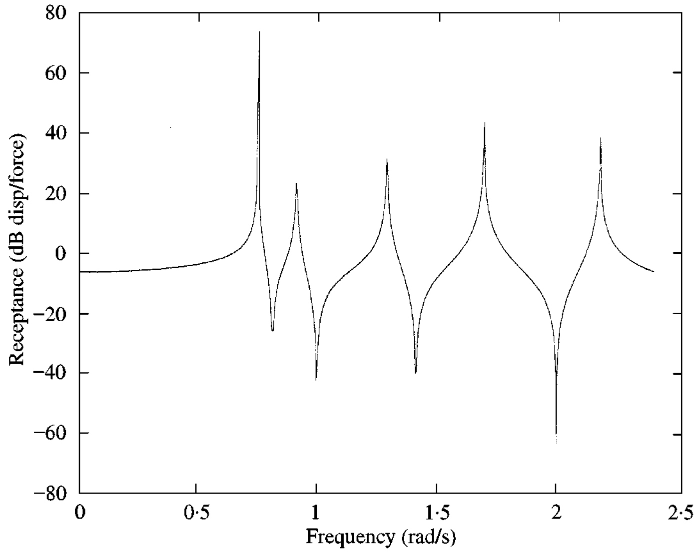

Figure 2. Frequency response h_{44}.

When the poles are coincident with zero $\bar{\lambda}_s$ to produce a node at the first co-ordinate $\boldsymbol{\beta}$ must be selected so that

$$(\varphi_{1r}\ \varphi_{1,r+1}, \ldots, \varphi_{1,r+p}]\,\boldsymbol{\beta} = 0. \tag{24}$$

To create a node at a different co-ordinate a different combination of the vectors would be needed.

$$(\varphi_{jr}\ \varphi_{j,r+1}, \ldots, \varphi_{j,r+p}]\,\boldsymbol{\gamma} = 0, \quad j \neq 1, \tag{25}$$

so that in principle it would be possible to produce a node at every co-ordinate of the system at the same frequency. The necessary and sufficient conditions established in reference [1] would hold for the case of a distinct zero. In the case of

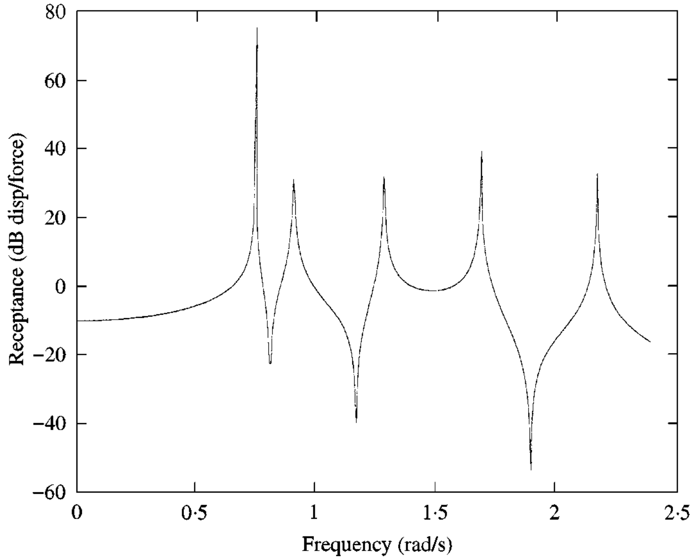

Figure 3. Frequency response h_{34}.

both repeated poles and repeated zeros, at the same frequency, the necessary and sufficient conditions in equations (21) and (22) would hold. Since the eigenvectors $\boldsymbol{\psi}_s \boldsymbol{\psi}_{s+1}, \ldots, \boldsymbol{\psi}_{s+m}$ are independent it is apparent that in general the $\boldsymbol{\alpha}$ which satisfies equation (21) is not unique. Therefore, it is generally possible for two (or more) poles to be cancelled by the same number of zeros to produce coincident vibration nodes at the same co-ordinate. It is shown in Appendix A that there are three cases which include all circumstances of repeated poles and repeated zeros at the same frequency. Specifically, there must be one more pole than the number of zeros, one more zero than the number of poles, or equal numbers of poles and zeros at every co-ordinate. In every case equation (21) is satisfied so that every cancellation produces a vibration node. The interlacing rules will not allow two repeated poles in a point receptance without there being a zero at the same frequency. Since a cancellation always creates a node (so that the zero is present in the point receptance and all the cross receptances of the same co-ordinate), it follows that whenever there are two repeated poles there will be a pole–zero cancellation at all co-ordinates at the same frequency.

4.1. NUMERICAL EXAMPLE

When, in Figure 1 $m_2 = 1{\cdot}388$, $m_4 = 2{\cdot}951$ and a stiffness $k_{10} = 0{\cdot}644$ is introduced between m_2 and m_4, and all the other masses and stiffness are unity two repeated poles occur at 0·3 Hz and a different coincident zero appears in every point receptance. The results are summarized in Table 3. The eigenvectors of the repeated poles are given in Table 4 from which it can be seen that a vector $\beta = (0{\cdot}9975,\ 0{\cdot}0714)^{\mathrm{T}}$ will satisfy equation (24) to give a vibration node at m_1. Different combinations of the vectors $\boldsymbol{\varphi}_5$ and $\boldsymbol{\varphi}_6$ are required to produce the

TABLE 3

Table of poles and zeros (Hz)

	Zeros					
Poles	h_{11}	h_{22}	h_{33}	h_{44}	h_{55}	h_{66}
0·0991	0·1023	0·1095	0·1125	0·1218	0·1205	0·1133
0·1332	0·1383	0·1542	0·1653	0·1648	0·1592	0·1912
0·1936	0·2206	0·2251	0·1944	0·2251	0·1955	0·2338
0·2353	0·2855	0·2398	0·2863	0·2941	0·2374	0·2847
0·3000	0·3000	0·3000	0·3000	0·3000	0·3000	0·3000
0·3000						

TABLE 4

Eigenvectors of the coincident poles

Eigenvectors	
$\boldsymbol{\varphi}_5$	$\boldsymbol{\varphi}_6$
– 0·0337	0·4708
0·0524	– 0·7311
– 0·1284	0·4517
0·1471	0·0297
– 0·9114	– 0·1837
0·3570	0·0720

vibration nodes at the other co-ordinates. When the zero term created by combining $\boldsymbol{\varphi}_5$ and $\boldsymbol{\varphi}_6$ is omitted one obtains the eigenvector of a coincident zero of the point and cross-receptances at the same co-ordinate. This point is illustrated in Figures 4–6. In Figures 4 and 5, all six point receptances are plotted and since there are only five peaks in each plot it is apparent that a pole–zero cancellation has occurred at each co-ordinate. Figure 6 shows that the same cancellation occurs in the cross-receptances. This means that there is a vibration node of a normal mode at 0·3 Hz at every co-ordinate.

5. CONCLUSIONS

Zeros generally occur at different frequencies in different frequency response measurements. But any frequency response that includes the co-ordinate of a node of a normal mode of vibration (either as the driving point or the measured point) will not contain any contribution from the mode. Therefore, for a pole and zero to cancel and produce a vibration node the zero must be present in the point receptance and all the cross receptances at the co-ordinate of the node. To create a node in this way there must be a coincident (same eigenvalue) pole and zero and

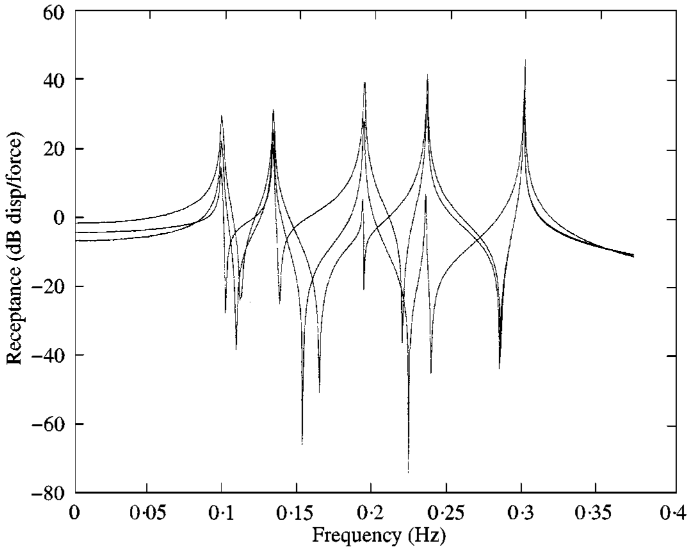

Figure 4. Point receptances h_{11}, h_{22} and h_{33}.

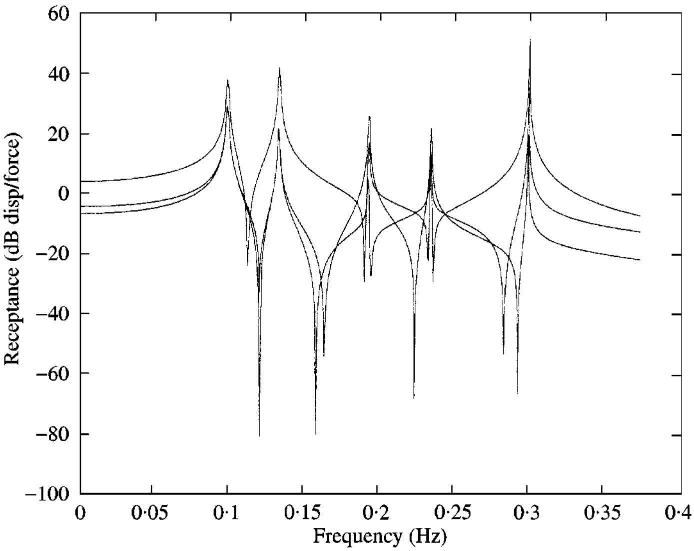

Figure 5. Point receptances h_{44}, h_{55} and h_{66}.

the eigenvector of the pole (excluding the nodal co-ordinate) must be identical to eigenvector of the zero. A pole and zero cannot cancel in any other way than to produce a node, although they may coexist at an identical frequency when there are either repeated poles or repeated zeros (or both). Different combinations of the eigenvectors of a repeated pole will always combine so that a cancellation with a different zero at every co-ordinate will create a vibration node at every co-ordinate at the same frequency.

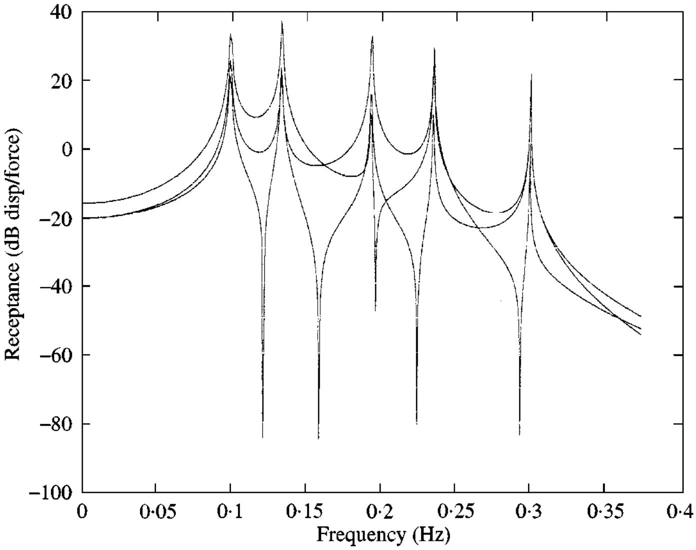

Figure 6. Cross-receptances h_{14}, h_{25} and h_{36}.

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REFERENCES

1. J. E. Mottershead and G. Lallement 1999 *Journal of Sound and Vibration* **222**, 833–851. Vibration nodes, and the cancellation of poles and zeros by unit-rank modifications to structures.
2. J. E. Mottershead 1998 *Mechanical Systems and Signal Processing*, **12**, 591–597. On the zeros of structural frequency response functions and their sensitivities.
3. R. J. Pomazal and V. C. Snyder 1971 *American Institute of Aeronautics and Astronautics Journal* **9**, 2216–2221. Local modifications of damped linear systems.
4. Q. Zhang and G. Lallement 1989 *Mechanical Systems and Signal Processing*, **3**, 55–69. Selective structural modifications: applications to the problems of eigensolution sensitivity and model adjustment.
5. J. He and Y. Q. Li 1994 *Modal Analysis* **10**, 224–235. Relocation of antiresonances of a vibratory system by local structural changes.
6. Y. Q. Li, J. He and G. Lleonart *Asia-Pacific Vibration Conference, Kitakyushu*, 1300–1306. Relocation of resonances and antiresonances via local structural modification.
7. E. K. L. Yee and Y. G. Tsuei 1991 *American Institute of Aeronautics and Astronautics Journal* **29**, 1973–1977. Method of shifting natural frequencies of damped mechanical systems.
8. J. E. Mottershead and M. I. Friswell 1993 *Journal of Sound and Vibration* **162**, 347–375. Model updating in structural dynamics: a survey.
9. M. I. Friswell and J. E. Mottershead 1995 *Finite Element Model Updating in Structural Dynamics*. Dordrecht: Kluwer Academic Publishers.

10. P. D. CHA and C. PIERRE 1999 *Journal of Sound and Vibration* **219**, 669–687. Imposing nodes to the normal modes of a linear elastic structure.
11. Y. M. RAM and S. ELHAY 1996 *Journal of Sound and Vibration* **195**, 607–615. The theory of a multi-degree-of-freedom dynamic absorber.
12. Y. M. RAM 1998 *Journal of Vibration and Control* **4**, 145–165. Pole–zero assignment of vibratory systems by state feedback control.
13. Y. M. RAM 1998 *The Quarterly Journal of Mechanics and Applied Mathematics* **51**, 461–476. Pole assignment for the vibrating rod.

APPENDIX A: REPEATED POLES AND REPEATED ZEROS AT THE SAME FREQUENCY

$\boldsymbol{\varphi}_r\,\boldsymbol{\varphi}_{r+1},\ldots,\boldsymbol{\varphi}_{r+p}$ are independent vectors which span $null[\mathbf{K}-\lambda_r\mathbf{M}]$.
$\boldsymbol{\psi}_s\,\boldsymbol{\psi}_{s+1},\ldots,\boldsymbol{\psi}_{s+m}$ are independent vectors which span $null[\bar{\mathbf{K}}-\bar{\lambda}_s\bar{\mathbf{M}}]$, $\bar{\lambda}_s=\lambda_r$.

A.1. CASE (a): MORE POLES THAN ZEROS ($p > m$)

The columns of

$$\left[\begin{array}{c}\mathbf{0}\\ \hline \tilde{\boldsymbol{\psi}}_s\tilde{\boldsymbol{\psi}}_{s+1},\;\cdots\;,\tilde{\boldsymbol{\psi}}_{s+(p-1)}\end{array}\right]$$

are p independent vectors formed from linear combinations of $\boldsymbol{\varphi}_r\,\boldsymbol{\varphi}_{r+1},\ldots,\boldsymbol{\varphi}_{r+p}$ with the constraint that the first term in each vector is zero. The choice of the first term is consistent with the analysis elsewhere in the paper and does not incur any loss of generality. The vectors $\tilde{\boldsymbol{\psi}}_s\tilde{\boldsymbol{\psi}}_{s+1},\ldots,\tilde{\boldsymbol{\psi}}_{s+(p-1)}$ all satisfy the condition

$$(\bar{\mathbf{k}}-\bar{\lambda}_s\bar{\mathbf{m}})^{\mathrm{T}}\,\tilde{\boldsymbol{\psi}}_i=0,\quad i=s,s+1,\ldots,s+(p-1),$$

and they are also eigenvectors of the repeated zeros. Thus, each $\tilde{\boldsymbol{\psi}}_i$ must be an independent linear combination of the vectors $\boldsymbol{\psi}_s\boldsymbol{\psi}_{s+1},\ldots,\boldsymbol{\psi}_{s+m}$ and $m=p-1$. This means that if there are fewer zeros than poles only one pole remains uncancelled by the zeros.

A.2. CASE (b): MORE ZEROS THAN POLES ($m > p$)

The vectors $\tilde{\boldsymbol{\psi}}_r\tilde{\boldsymbol{\psi}}_{r+1},\ldots,\tilde{\boldsymbol{\psi}}_{r+(m-1)}$ can be formed from linear combinations of $\boldsymbol{\psi}_s\boldsymbol{\psi}_{s+1},\ldots,\boldsymbol{\psi}_{s+m}$ with the contraint that,

$$(\bar{\mathbf{k}}-\bar{\lambda}_s\bar{\mathbf{m}})^{\mathrm{T}}\,\tilde{\boldsymbol{\psi}}_i=\mathbf{0},\quad i=r,r+1,\ldots,r+(m-1),$$

It follows that

$$\left[\begin{array}{c}\mathbf{0}\\ \hline \tilde{\boldsymbol{\psi}}_r\tilde{\boldsymbol{\psi}}_{r+1},\;\cdots\;,\tilde{\boldsymbol{\psi}}_{r+(m-1)}\end{array}\right]$$

contains in its columns the eigenvectors of the repeated poles and that $p=m-1$.

A.3. CASE (c): EQUAL NUMBERS OF REPEATED POLES AND ZEROS ($p = m$).

This is the case when

$$[\boldsymbol{\varphi}_r\, \boldsymbol{\varphi}_{r+1}, \ldots, \boldsymbol{\varphi}_{r+p}] = \left[\begin{array}{c} \mathbf{0} \\ \hdashline \tilde{\boldsymbol{\psi}}_s\, \tilde{\boldsymbol{\psi}}_{s+1}, \; \cdots \; , \tilde{\boldsymbol{\psi}}_{s+m} \end{array} \right]$$

so that a node is produced at the first co-ordinate in the eigenvector of every repeated pole. This is the only available result when $m \neq p - 1$ and $p \neq m - 1$. When it occurs the repeated poles cancel with all the repeated zeros.